

A short proof of the zero-two law for cosine functions

Jean Esterle

Abstract : Let $(C(t))_{t \in \mathbb{R}}$ be a cosine function in a unital Banach algebra. We give a simple proof of the fact that if $\limsup_{t \rightarrow 0} \|C(t) - 1_A\| < 2$, then $\limsup_{t \rightarrow 0} \|C(t) - 1_A\| = 0$.

Keywords : Cosine function, scalar cosine function, commutative local Banach algebra.

AMS classification : Primary 46J45, 47D09, Secondary 26A99

1 Introduction

Recall that a cosine function taking values in a unital normed algebra A with unit element 1_A is a family $C = (C(t))_{t \in \mathbb{R}}$ of elements of A satisfying the so-called d'Alembert equation

$$C(0) = 1_A, C(s+t) + C(s-t) = 2C(s)C(t) \quad (s \in \mathbb{R}, t \in \mathbb{R}). \quad (1)$$

One can define in a similar way cosine sequences $(C(n))_{n \in \mathbb{Z}}$. A cosine sequence depends only on the value of $C(1)$, since we have, for $n \geq 1$,

$$C(-n) = C(n) = T_n(C(1)),$$

where $T_n(x) = \sum_{k=0}^{[n/2]} C_n^k x^{n-2k} (x^2 - 1)^k$ is the n^{th} -Tchebyshev polynomial.

Strongly continuous operator valued cosine functions play an important role in the study of abstract nonlinear second order differential equations, see for example [11]. In a paper to appear in the Journal of Evolution Equations [9], Schwenninger and Zwart showed that if a strongly continuous cosine family $(C(t))_{t \in \mathbb{R}}$ of bounded operators on a Banach space X satisfies $\limsup_{t \rightarrow 0} \|C(t) - I_X\| < 2$, then the generator a of this cosine function is a bounded operator, so that $\limsup_{t \rightarrow 0} \|C(t) - I_X\| = \limsup_{t \rightarrow 0} \|\cos(ta)I_X - I_X\| = 0$, and they asked whether a similar zero-two law holds for general cosine functions $(C(t))_{t \in \mathbb{R}}$. This

question was answered positively by Chojnacki in [5]. Using a sophisticated argument based on ultrapowers, Chonajcki deduced this zero-two law from the fact that if a cosine sequence $C(t)$ satisfies $\sup_{t \in \mathbb{R}} \|C(t) - 1_A\| < 2$, then $C(t) = 1_A$ for $t \in \mathbb{R}$. This second result, which was obtained independently by the author in [7], is proved by Chojnacki in [5] by adapting methods used by Bobrowski, Chojnacki and Gregosiewicz in [3] to show that if a cosine sequence $(C(t))_{t \in \mathbb{R}}$ satisfies $\sup_{t \in \mathbb{R}} \|C(t) - \cos(at)1_A\| < \frac{8}{3\sqrt{3}}$ for some $a \in \mathbb{R}$, then $C(t) = \cos(at)1_A$ for $t \in \mathbb{R}$, a result also obtained independently by the author in [7], which improves previous results of [2], [4] and [10].

The purpose of this paper is to give a short direct proof of the zero-two law. The zero-two law for complex-valued cosine functions is a folklore result, which easily implies that if $\limsup_{t \rightarrow 0} \rho(C(t) - 1_A) < 2$ then $\limsup_{t \rightarrow 0} \rho(C(t) - 1_A) = 0$, where $\rho(x)$ denotes the spectral radius of an element x of a Banach algebra, see section 2. Our proof of the zero-two law is then based on the fact that if $\|C(t) - 1_A\| \leq 2$, and if $\rho\left(C\left(\frac{t}{2}\right) - 1_A\right) < 1$, then we have

$$C\left(\frac{t}{2}\right) = \sqrt{1_A - \frac{C(t) - 1_A}{2}},$$

where $\sqrt{1_A - u}$ is defined by the usual series for $\|u\| \leq 1$. It follows from this identity and from the fact that the coefficients of the Taylor series at the origin of the function $t \rightarrow 1 - \sqrt{1 - t}$ are positive that in this situation we have

$$\left\|C\left(\frac{t}{2}\right) - 1_A\right\| \leq 1 - \sqrt{1 - \left\|\frac{C(t) - 1_A}{2}\right\|},$$

and the zero-two law follows.

Notice that if we replace the constant 2 by $\frac{3}{2}$ a "three line argument" due to Arendt [1] shows that if $\limsup_{t \rightarrow 0} \|C(t) - 1_A\| < \frac{3}{2}$ then $\limsup_{t \rightarrow 0} \|C(t) - 1_A\| = 0$. The proof presented here has some analogy with Arendt's proof, and the difficulty to estimate $\|(1_A + C(\frac{t}{2}))^{-1}\|$ is circumvented in the present paper by using the formula above.

The author wishes to thank W. Chonajcki for giving information about the reference [5]. He also thanks F. Schwenninger for valuable discussions about the content of the paper.

2 The zero-two law for the spectral radius

The zero-two law for scalar cosine functions pertains to folklore, but we could not find a reference in the literature for the following certainly well-known lemma, which is a variant of proposition 3.1 of [7].

Lemma 2.1. *Let $c = (c(t))_{t \in \mathbb{R}}$ be a complex-valued cosine function. Then c satisfies one of the following conditions*

- (i) $\limsup_{t \rightarrow 0} |c(t) - 1| = +\infty$,
- (ii) $\limsup_{t \rightarrow 0} |c(t) - 1| = 2$,
- (iii) $\limsup_{t \rightarrow 0} |c(t) - 1| = 0$.

First assume that $M := \limsup_{t \rightarrow 0} |c(t)| < +\infty$, and denote by S the set of all complex numbers α for which there exists a sequence $(t_m)_{m \geq 1}$ of positive real numbers such that $\lim_{m \rightarrow +\infty} t_m = 0$ and $\lim_{m \rightarrow +\infty} c(t_m) = \alpha$. Then $|\alpha| \leq M$ for every $\alpha \in S$. Notice that if $\alpha \in S$, and if a sequence $(t_m)_{m \in \mathbb{Z}}$ satisfies the above conditions with respect to α , then $T_n(\alpha) = \lim_{m \rightarrow +\infty} T_n(C(t_m)) = \lim_{m \rightarrow +\infty} C(nt_m)$, and so $T_n(\alpha) \in S$, and $|T_n(\alpha)| \leq M$ for $n \geq 1$. Now write $\alpha = \cos(z) = \sum_{k=0}^{+\infty} \frac{z^{2k}}{(2k)!}$, and set $u = \operatorname{Re}(z)$, $v = \operatorname{Im}(z)$. We have, for $n \geq 1$,

$$T_n(\alpha) = \cos(nz) = \frac{e^{inu} e^{-nv} + e^{-inu} e^{nv}}{2}.$$

Since $\sup_{n \geq 1} |T_n(\alpha)| \leq M$, we have $v = 0$, $S \subset [-1, 1]$, and $\limsup_{t \rightarrow 0} |c(t) - 1| \leq 2$.

Now assume that $S \neq \{1\}$, and let $\alpha \in S \setminus \{1\}$. We have $\alpha = \cos(u)$ for some $u \in \mathbb{R}$. We see as above that $\cos(nu) \in S$ for every $n \geq 1$. If u/π is irrational, then the set $\{e^{inu}\}_{n \geq 1}$ is dense in the unit circle \mathbb{T} , and so $S = [-1, 1]$ since S is closed, and in this situation $\limsup_{t \rightarrow 0} |c(t) - 1| = 2$.

Now assume that u/π is rational, and let $s \geq 2$ be the smallest positive integer such that $e^{ius} = 1$. Then $e^{\frac{2i\pi}{s}} = e^{ipu}$ for some positive integer p , and so $\cos(\frac{2\pi}{s}) \in S$. Let $(t_m)_{m \geq 1}$ be a sequence of positive reals such that $\lim_{m \rightarrow +\infty} t_m = 0$ and $\lim_{m \rightarrow +\infty} c(t_m) = \cos(\frac{2\pi}{s})$, let $q \geq 2$, and let β be a limit point of the sequence $c(\frac{t_m}{s^{q-1}})_{m \geq 1}$. There exists $y \in \mathbb{R}$ such that $\cos(y) = \beta$ and such that $s^{q-1}y = \frac{2\pi}{s} + 2k\pi$, with $k \in \mathbb{Z}$. Then $y = (1 + ks)\frac{2\pi}{s^q}$. Since $\gcd(1 + ks, s^q) = 1$, there exists a positive integer r such that $ry - \frac{2\pi}{s^q} \in 2\pi\mathbb{Z}$, so that $\cos(\frac{2\pi}{s^q}) \in S$. This implies that $\cos(\frac{2p\pi}{s^q}) \in S$ for $p \geq 1$, $q \geq 1$, and S is dense in $[-1, 1]$. Since S is closed, we obtain again $S = [-1, 1]$, which implies that $\limsup_{t \rightarrow 0} |c(t) - 1| = 2$. So if neither (i) nor (ii) holds, we have $S = \{1\}$, which implies (iii). \square

Notice that if a cosine function $C = (C(t))_{t \in \mathbb{R}}$ in a Banach algebra A satisfies $\sup_{|t| \leq \eta} \|C(t)\| \leq M < +\infty$ for some $\eta > 0$, then $\sup_{|t| \leq L} \|C(t)\| < +\infty$ for every $L > 0$, since $\sup_{|t| \leq n\eta} \|C(t)\| \leq \sup_{\|y\| \leq M} \|T_n(y)\|$ for every $n \geq 1$, where T_n denotes the n -th Tchebyshev polynomial. In particular if a complex-valued cosine function $c = (c(t))_{t \in \mathbb{R}}$ satisfies (iii), then the identity

$$(1 - c(s - t))(1 - c(s + t)) = (c(s) - c(t))^2$$

shows as is well-known that the cosine function c is continuous on \mathbb{R} , which implies that $c(t) = \cos(ta)$ for some $a \in \mathbb{C}$.

If A is commutative and unital, we will denote 1_A the unit element of A , and we will denote by \widehat{A} the space of all characters on A , equipped with the Gelfand topology, i.e. the compact topology induced by the weak* topology on the unit ball of the dual space of A .

Proposition 2.2. *Let $C = (C(t))$ be a cosine function in a unital Banach algebra A . If $\limsup_{t \rightarrow 0} \rho(C(t) - 1_A) < 2$, then $\limsup_{t \rightarrow 0} \rho(C(t) - 1_A) = 0$.*

Proof: We may assume that unital Banach algebra A is generated by $(C(t))_{t \in \mathbb{R}}$. Let $\chi \in \widehat{A}$. Then the cosine complex-valued function $(\chi(C(t)))_{t \in \mathbb{R}}$ satisfies condition (iii) of the lemma, and so there exists $a_\chi \in \mathbb{C}$ such that we have

$$\chi(C(t)) = \cos(ta_\chi) \quad (t \in \mathbb{R}).$$

Set $u_\chi = \operatorname{Re}(a_\chi)$, $v_\chi = \operatorname{Im}(a_\chi)$. We have

$$\rho(C(t) - 1) \geq |1 - \cos(tu_\chi) \operatorname{ch}(tv_\chi)|.$$

If the family $(u_\chi)_{\chi \in \widehat{A}}$ were unbounded, there would exist a sequence $(t_n)_{n \geq 1}$ of real numbers converging to zero and a sequence $(\chi_n)_{n \geq 1}$ of characters of A such that $\cos(t_n u_{\chi_n}) = -1$, and we would have $\rho(C(t_n) - 1) \geq 2$ for $n \geq 1$. So the family $(u_\chi)_{\chi \in \widehat{A}}$ is bounded. If the family $(v_\chi)_{\chi \in \widehat{A}}$ were unbounded, there would exist a sequence $(t'_n)_{n \geq 1}$ of real numbers converging to zero and a sequence $(\chi'_n)_{n \geq 1}$ of characters of A such that $\lim_{n \rightarrow +\infty} \operatorname{ch}(t'_n v_{\chi'_n}) = +\infty$. But this would imply that $\limsup_{t \rightarrow 0} \rho(C(t)) = +\infty$. Hence the family $(a_\chi)_{\chi \in \widehat{A}}$ is bounded, and we have

$$\limsup_{t \rightarrow 0} \rho(C(t) - 1_A) = \lim_{t \rightarrow 0} \sup_{\chi \in \widehat{A}} |\cos(ta_\chi) - 1| = 0.$$

□

3 The zero-two law for cosine functions

Set $\alpha_n = \frac{1}{n!} \frac{1}{2} \left(\frac{1}{2} - 1\right) \dots \left(\frac{1}{2} - n + 1\right)$ for $n \geq 1$, with the convention $\alpha_0 = 0$, and for $|z| < 1$, set

$$\sqrt{1-z} = \sum_{n=0}^{+\infty} (-1)^n \alpha_n z^n,$$

so that $(\sqrt{1-z})^2 = 1-z$, and $\sqrt{1-t}$ is the positive square root of $1-t$ for $t \in (-1, 1)$. Also $\operatorname{Re}(\sqrt{1-z}) > 0$ for $|z| < 1$.

We have, for $t \in [0, 1)$,

$$\sum_{n=1}^{+\infty} (-1)^{n-1} \alpha_n t^n = 1 - \sqrt{1-t}.$$

Since $(-1)^{n-1} \alpha_n \geq 0$ for $n \geq 1$, the series $\sum_{n=1}^{+\infty} |\alpha_n| = \sum_{n=1}^{+\infty} (-1)^{n-1} \alpha_n$ is convergent, and we have

$$\sum_{n=1}^{+\infty} |\alpha_n| t^n = 1 - \sqrt{1-t} \quad (0 \leq t \leq 1).$$

Now let A be a commutative unital Banach algebra, and let $x \in A$ such that $\|x\| \leq 1$. Set

$$\sqrt{1_A - x} = \sum_{n=0}^{+\infty} (-1)^n \alpha_n x^n.$$

Then $(\sqrt{1_A - x})^2 = 1_A - x$, and we have

$$\|1_A - \sqrt{1_A - x}\| = \left\| \sum_{n=1}^{+\infty} (-1)^n \alpha_n x^n \right\| \leq \sum_{n=1}^{+\infty} |\alpha_n| \|x\|^n = 1 - \sqrt{1 - \|x\|} \quad (2)$$

Notice also that if A is commutative, then we have

$$\operatorname{Re} \left(\chi \left(\sqrt{1_A - x} \right) \right) = \operatorname{Re} \left(\sqrt{1 - \chi(x)} \right) \geq 0 \quad (\chi \in \hat{A}). \quad (3)$$

We obtain the following formula

Lemma 3.1. *Let $(C(t))_{t \in \mathbb{R}}$ be a cosine function in a unital Banach algebra A . Assume that $\|C(t) - 1_A\| \leq 2$ and that $\rho((C(\frac{t}{2}) - 1) < 1$. Then we have*

$$C\left(\frac{t}{2}\right) = \sqrt{1_A - \frac{1_A - C(t)}{2}}.$$

Proof : The abstract version of the formula $\sin^2\left(\frac{u}{2}\right) = \frac{1 - \cos(u)}{2}$ gives

$$1_A - C\left(\frac{t}{2}\right)^2 = \frac{1_A - C(t)}{2}, C\left(\frac{t}{2}\right)^2 = 1_A - \frac{1_A - C(t)}{2} = \left(\sqrt{1_A - \frac{1_A - C(t)}{2}} \right)^2,$$

$$\left(C\left(\frac{t}{2}\right) - \sqrt{1_A - \frac{1_A - C(t)}{2}} \right) \left(C\left(\frac{t}{2}\right) + \sqrt{1_A - \frac{1_A - C(t)}{2}} \right) = 0.$$

We may assume that A is commutative. Let $\chi \in \hat{A}$. Since $\rho(C(\frac{t}{2}) - 1_A) < 1$, we have $\operatorname{Re}(\chi(C(\frac{t}{2}))) > 0$. Since $\operatorname{Re}\left(\chi\left(\sqrt{1_A - \frac{1_A - C(t)}{2}}\right)\right) \geq 0$, $C(\frac{t}{2}) + \sqrt{1_A - \frac{1_A - C(t)}{2}}$ is invertible in A , and $C(\frac{t}{2}) - \sqrt{1_A - \frac{1_A - C(t)}{2}} = 0$. \square

Theorem 3.2. *Let $(C(t))_{t \in \mathbb{R}}$ be a cosine sequence in a Banach algebra. If $\limsup_{t \rightarrow 0} \|C(t) - 1_A\| < 2$, then $\limsup_{t \rightarrow 0} \|C(t) - 1_A\| = 0$.*

Proof : It follows from proposition 2.2 and lemma 3.1 that there exists $\eta > 0$ such that we have, for $|t| \leq \eta$,

$$\|C(t) - 1_A\| < 2, C\left(\frac{t}{2}\right) = \sqrt{1_A - \frac{1_A - C(t)}{2}}.$$

Using (1), we obtain, for $|t| \leq \eta$,

$$\left\|C\left(\frac{t}{2}\right) - 1_A\right\| \leq 1 - \sqrt{1 - \left\|\frac{C(t) - 1_A}{2}\right\|}.$$

Set $l = \limsup_{t \rightarrow 0} \|C(t) - 1_A\|$. We obtain

$$l \leq 1 - \sqrt{1 - \frac{l}{2}} \leq 1,$$

and so $l = 0$. \square

Notice that the proof above gives a little bit more than the zero-two law : if $\|1 - C(t)\| \leq 2$ and $\rho(1 - C(\frac{t}{2})) < 1$ for $|t| \leq \eta$, then we have, for $n \geq 1$,

$$\sup_{|t| \leq 2^{-n}\eta} \|C(t) - 1_A\| \leq u_n,$$

where the sequence u_n satisfies $u_0 = 2, u_{n+1} = 1 - \sqrt{1 - \frac{u_n}{2}}$ for $n \geq 1$, and $\lim_{n \rightarrow +\infty} u_n = 0$, which gives an explicit control on the convergence to 0 of $\|C(t) - 1_A\|$ as $t \rightarrow 0$.

Notice also that the fact that the coefficients of the Taylor expansion at the origin of the function $t \rightarrow 1 - \sqrt{1 - t}$ are positive was used in [6] to show that $\|x^2 - x\| \geq 1/4$ for every quasinilpotent element x of a Banach algebra such that $|x| \geq 1/2$. Similar argument were used in [8] to show that if a semigroup $(T(t))_{t \geq 0}$ in a Banach algebra A satisfies $\limsup_{t \rightarrow 0^+} \|T(t) - T((n+1)t)\| < \frac{n}{(n+1)^{1+\frac{1}{n}}}$ for some $n \geq 1$, then there exists an idempotent J of A such that $\lim_{t \rightarrow 0} \|T(t) - J\| = 0$, so that $\limsup_{t \rightarrow 0^+} \|T(t) - T((n+1)t)\| = 0$.

Références

- [1] W. Arendt, *A $0-3/2$ - Law for Cosine Functions*. Ulmer Seminare, Funktionalanalyse und Evolutionsgleichungen, 17 (2012), 349-350.
- [2] A. Bobrowski and W. Chojnacki, *Isolated points of some sets of bounded cosine families, bounded semigroups, and bounded groups on a Banach space*, Studia Math. 217 (2013), 219-241.
- [3] A. Bobrowski, W. Chojnacki, and A. Gregosiewicz, *On close-to-scalar one-parameter cosine families*, J. Math. Anal. Appl. (2015), <http://dx.doi.org/10.1016/j.jmaa.2015.04.012>.
- [4] W. Chojnacki, *On cosine families close to scalar cosine families*, to appear in J. Aust. Math. Soc., arXiv :1411.0854.
- [5] W. Chojnacki, *Around Schwenninger and Zwart's zero-two law for cosine families*, submitted.
- [6] J. Esterle, *Quasimultipliers, representations of H^∞ , and the closed ideal problem for commutative Banach algebras*, Springer Lect. Notes 975 (1983), 66-162.
- [7] J. Esterle, *Bounded cosine functions close to continuous bounded scalar cosine functions*, submitted. <https://hal.archives-ouvertes.fr/hal-01111839>
- [8] J. Esterle and A. Mokhtari, *Distance entre éléments d'un semi-groupe dans une algèbre de Banach*, J. Func. An 195 (2002), 167-189.
- [9] F. Schwenninger and H. Zwart, *Zero-two laws for cosine families*, J. Evolution Eq., to appear, arXiv :1402.1304v3.
- [10] F. Schwenninger and H. Zwart, *Less than one, implies 0*, submitted.
- [11] C. Travis and G. Webb, *Cosine families and abstract nonlinear second order differential equation*, Acta Math. Acad. Sci. Hungar 32(1978), 75-96.

Jean Esterle
IMB, UMR 5251
Université de Bordeaux
351, cours de la Libération
33405 - Talence (France)
esterle@math.u-bordeaux1.fr